Generalized strength of weighted scale-free networks

S. Furuya* and K. Yakubo†
Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan
(Received 19 February 2008; revised manuscript received 20 October 2008; published 5 December 2008)

An analysis to describe statistical properties of weighted complex networks is proposed. Effective structures of weighted networks depend on how strongly weights w are paid attention or which weights are relevant to the network problem. Defining the metaweight w^q with a real parameter q, we characterize systematically weighted complex networks depending on the level of importance of weights. It is found that power-law distribution functions $R_q[s(q)]$ of metastrengths $s(q)$ defined by $s_i(q)=\Sigma_j w_{ij}^q$, where i and j denote node indices for any q are characterized by only three exponents if the weight distribution is independent of network topology. We also examine the validity of our analytical arguments and the meaning of power-law forms of $R_q[s(q)]$ for different q values by illustrating some examples.

DOI: 10.1103/PhysRevE.78.066104 PACS number(s): 89.75.Hc, 89.75.Fb, 05.10.Gg

I. INTRODUCTION

Networks with more complicated topology than regular or disordered lattices have been extensively studied to describe a wide range of complex systems, such as communication networks [1–3], biological networks [4,5], collaboration relationships [6–8], transportation systems [6,9], ecological networks [10–13], and networks in condensed matter [14,15]. Empirical evidences show that most of these real-world networks share several common properties, namely, the small-world [16], scale-free [17], and fractal natures [18]. In particular, the scale-free property defined as a power-law behavior of the degree distribution $P(k)\propto k^{-\gamma_k}$ is of crucial importance because the degree exponent γ_k governs quantitative behaviors of physical quantities on a scale-free network. These features reflect topological aspects of networks, in which all nodes and edges are treated as depersonalized elements. In most of real-world networks, however, nodes and edges are not equivalent and multivalued quantities (weights) are assigned on them. Recently, much attention has been paid to networks with weighted edges, because properties of networks are deeply related to how weights distribute in networks [6,19–25]. In these works, various quantities defined in unweighted networks have been naturally extended for weighted networks. For example, the degree k_i of the node i in an unweighted network is extended to the strength $s_i=\Sigma_j w_{ij}$, where w_{ij} is the weight assigned to the edge (i,j). It has been found that many of weighted complex networks have power-law distributions of weights $[Q(w)\propto w^{-\gamma_w}]$ and strengths $[R(s)\propto s^{-\gamma_s}]$ [6,20]. The power-law strength distribution implies that most of real networks exhibit the scale-free property even in a sense of weighted networks.

In previous works on weighted networks, interests have been concentrated on connectivity of nodes and edges with large values of w and s. This is, however, not sufficient to describe the whole nature of weighted networks. It would be also important to know how weak edges (with small weights) construct the network. Inversely, in some cases, we might need to study the network structure with emphasizing edges with large weights more strongly than done by weights themselves. In a trading network of companies, for example, the network structure comprised of small companies connected to each other by weak edges is crucial because the entire economy strongly depends on numerically major small companies. Another example is a power grid network whose nodes and edges are generators or transformers and transmission lines, respectively. Regarding currents I on edges as weights, we can obtain information about statistical properties of the current network. When the nature of the power-supply network is required, however, edges with large weights should be more strongly emphasized than what is done by weights I, because the power is proportional to I^2. Therefore, statistical properties of a weighted network depend on how strongly we pay attention to weights.

In this paper, we propose a method to analyze the intrinsic nature of weighted complex networks. By introducing the metaweight defined by w_{ij}^q, where q is a real parameter, we characterize systematically structural features of networks depending on how strongly weights are emphasized. We show that only three exponents are required to characterize power-law distribution functions of metastrengths defined in the next section for any q if the weight distribution is independent of network topology. Namely, in such a case, the exponent characterizing the metastrength distribution function for arbitrary value of q can be obtained by these three exponents. Furthermore, the validity of our analytical arguments and the meaning of power-law forms of metastrength distributions for different q values are examined by illustrating some examples.

This paper is organized as follows. In Sec. II, we give definitions of the metaweight and the metastrength. It is also shown in this section that only three independent exponents can characterize the network in the case that the weight distribution is not related to network topology. We present examples of metaweight analyses in Sec. III by demonstrating several models of weighted complex networks and a real-world network of stock price correlations. Finally, the conclusions are given in Sec. IV. The application of the metaweight analysis to the weighted scale-free property, treated in

*s-furuya@eng.hokudai.ac.jp
†yakubo@eng.hokudai.ac.jp

this paper, is an example of this analysis. We emphasize wide applicability of the metaweight analysis in this section.

II. METAWEIGHT ANALYSIS

A. Metaweight and metastrength

In order that the scale-free property of a weighted complex network is more systematically studied by considering how significantly weighted edges contribute to the network structure, we introduce the metaweight defined by w^q, where w is an original weight assigned to an edge and q is a real parameter. For $|q| \sim 0$, weights in the network do not influence significantly statistical properties of the network, and we can concentrate on topological aspects of the network at $q=0$. Analyzing w^q with $q>1$ enables us to emphasize intense weights more strongly than done by original weights. On the contrary, for $q<0$, weak edges with small weights are emphasized. In the example of the trading network, a weighted structure dominated by small companies with weak edges can be evaluated by analyzing w^q with $q<0$. Therefore, it becomes possible by introducing metaweights to study statistical properties of inherent structures embedded in a given weighted network. Although statistical properties of powers of weights have been discussed for a specific networks model by Mukherjee and Manna [21], our arguments, differently from their paper, aim to reveal systematically the intrinsic nature of general weighted networks. The idea of the metaweight analysis seems to resemble that of the multifractal analysis [26] in which fractal dimensions of spatiality distributed measures are determined with paying attention to how significantly measures contribute to the distribution by introducing qth power of measures. Regardless of the apparent similarity, however, the metaweight analysis for weighted complex networks is largely different from the multifractal analysis as we will show below.

Among many statistical quantities characterizing weighted complex networks, in this paper, we discuss the distribution of metastrengths defined by

$$s_i(q) \equiv \sum_j w_{ij}^q, \tag{1}$$

where w_{ij} is the original weight assigned to the edge connecting the ith node to the jth. Here we assume that the weight w_{ij} is always positive. If the distribution function of metastrengths obeys a power law, the power-law exponent $\gamma_s(q)$ is a function of q. This q-dependent metastrength exponent $\gamma_s(q)$ characterizes the weighted scale-free network in more detail than described only by the conventional strength exponent γ_s.

B. Topology independent weight distribution

In order to clarify the functional form of the metastrength exponent $\gamma_s(q)$, we consider a situation that weights on edges are distributed independently of network topology. Applying the argument by Dorogovtsev and Mendes [27] to power-law metastrength distributions, we show that only three metastrength exponents for suitable q provide $\gamma_s(q)$ for any q.

Let us denote the distribution functions of degree k, metaweight w^q, and metastrength $s(q)$ by $P(k)$, $Q_q(w^q)$, and $R_q[s(q)]$, respectively. We assume that these distribution functions have power-law forms for large arguments, namely,

$$P(k) \propto k^{-\gamma_k} \quad (k \gg 1), \tag{2}$$

$$Q_q(w^q) \propto (w^q)^{-\gamma_w(q)} \quad (w^q \gg 1), \tag{3}$$

$$R_q[s(q)] \propto s(q)^{-\gamma_s(q)} \quad [s(q) \gg 1]. \tag{4}$$

Allowing divergent exponents, our conclusions hold even for exponential distribution functions. Generating functions of these distributions are defined by [28]

$$\tilde{X}(z) = \sum_x z^x X(x), \tag{5}$$

where X represents each of the distribution functions P, Q_q, and R_q. Assuming that the asymptotic form of $X(x)$ is proportional to $x^{-\gamma}$ for large x and replacing the summation in Eq. (5) by an integral, the generating function $\tilde{X}(z)$ has a term proportional to $(1-z)^{\gamma-1}$ [29]. It should be noted that this term is non-analytic at $z=1$ if the exponent γ is noninteger [30]. Therefore, from the asymptotic behaviors Eqs. (2)–(4), the generating functions can be expanded around $z=1$ as

$$\tilde{P}(z) = 1 - \sum_{n>0} a_n^P (1-z)^n - b^P (1-z)^{\gamma_k - 1}, \tag{6}$$

$$\tilde{Q}_q(z) = 1 - \sum_{n>0} a_n^{Q_q} (1-z)^n - b^{Q_q} (1-z)^{\gamma_w(q)-1}, \tag{7}$$

$$\tilde{R}_q(z) = 1 - \sum_{n>0} a_n^{R_q} (1-z)^n - b^{R_q} (1-z)^{\gamma_s(q)-1}. \tag{8}$$

Since the generating function of the sum of independent stochastic variables is generally the product of the generating functions of these variables [28], the generating function of the metastrength distribution is given by $\tilde{R}_q(z) = \sum_k [\tilde{Q}_q(z)]^k P(k)$ if stochastic variables w^q on edges are independent of each other. Thus, using Eq. (5), we have

$$\tilde{R}_q(z) = \tilde{P}[\tilde{Q}_q(z)]. \tag{9}$$

Substituting Eqs. (6)–(8) into Eq. (9), the right-hand side of Eq. (9) is

$$\tilde{P}[\tilde{Q}_q(z)] = 1 - \sum_{n>0} a_n^P \left[\sum_{m>0} a_m^{Q_q} (1-z)^m + b^{Q_q} (1-z)^{\gamma_w(q)-1} \right]^n - b^P \left[\sum_{m>0} a_m^{Q_q} (1-z)^m + b^{Q_q} (1-z)^{\gamma_w(q)-1} \right]^{\gamma_k - 1}. \tag{10}$$

Since the asymptotic behaviors of P, Q_q, and R_q are interested, only the analytic and nonanalytic leading terms of expansions around $z=1$ are argued here. Neglecting higher order nonanalytic terms than $O[(1-z)^{\gamma_w(q)-1}]$, the second term of the right-hand side of Eq. (10) becomes

$$\sum_{n>0} a_n^P\left[\sum_{m>0} a_m^{Q_q}(1-z)^m\right]^n + a_1^P b^{Q_q}(1-z)^{\gamma_w(q)-1}.$$

On the contrary, the third term of Eq. (10), which gives only nonanalytic terms, is expanded in different ways depending on the value of $\gamma_w(q)$. If $\gamma_w(q)>2$, $\Sigma_{m>0}a_m^{Q_q}(1-z)^m$ is larger than $b^{Q_q}(1-z)^{\gamma_w(q)-1}$ for $z\approx 1$ and the third term of Eq. (10) is expanded as

$$\text{third term} = b^P\left[\sum_{m>0} a_m^{Q_q}(1-z)^m\right]^{\gamma_k-1} \times \sum_{l=0}^{\infty}\frac{(\gamma_k-1)(\gamma_k-2)\cdots(\gamma_k-l)}{l!} \times \left[\frac{b^{Q_q}(1-z)^{\gamma_w(q)-1}}{\Sigma_{n>0}a_n^{Q_q}(1-z)^n}\right]^l.$$

Within the lowest order of $(1-z)$, this term is approximated by $a_1^{Q_q}b^P(1-z)^{\gamma_k-1}$, while the nonanalytic term of the second term of Eq. (10) is $a_1^P b^{Q_q}(1-z)^{\gamma_w(q)-1}$. Therefore, the lowest order nonanalytic term of $\widetilde{P}[\widetilde{Q}_q(z)]$ is proportional to $(1-z)^{\min[\gamma_k-1,\gamma_w(q)-1]}$. Comparing this with the nonanalytic term of Eq. (8), we have

$$\gamma_s(q) = \min[\gamma_k, \gamma_w(q)] \tag{11}$$

for $\gamma_w(q)>2$. On the other hand, in the case of $\gamma_w(q)<2$, the third term of Eq. (10) is expanded as

$$\text{third term} = b^P[b^{Q_q}(1-z)^{\gamma_w(q)-1}]^{\gamma_k-1} \times \sum_{l=0}^{\infty}\frac{(\gamma_k-1)(\gamma_k-2)\cdots(\gamma_k-l)}{l!} \times \left[\frac{\Sigma_{n>0}a_n^{Q_q}(1-z)^n}{b^{Q_q}(1-z)^{\gamma_w(q)-1}}\right]^l,$$

where the lowest order term is $b^{Q_q}b^P(1-z)^{(\gamma_k-1)[\gamma_w(q)-1]}$. The same argument with the above leads to

$$\gamma_s(q) = \min\{\gamma_w(q), (\gamma_k-1)[\gamma_w(q)-1]+1\}, \tag{12}$$

for $\gamma_w(q)<2$. At $\gamma_w(q)=2$, two relations (11) and (12) becomes equivalent, namely,

$$\gamma_s(q) = \min[\gamma_k, 2] \quad \text{at } \gamma_w(q)=2. \tag{13}$$

It should be noted that the relation $\gamma_s(q)=(\gamma_k-1)[\gamma_w(q)-1]+1$ holds only when $\gamma_w(q)<2$ and $\gamma_k<2$. This is because $\gamma_w(q)$ is always larger than unity due to the normalization condition of $Q_q(w^q)$ and the quantity $(\gamma_k-1)[\gamma_w(q)-1]+1-\gamma_w(q)=(\gamma_k-2)[\gamma_w(q)-1]$ can be negative only when $\gamma_k<2$. From the relations (11) and (12), we can say that the metastrength distribution $R_q[s(q)]$ obeys the power law with the exponent γ_k if the distribution $Q_q(w^q)$ decays exponentially $[\gamma_w(q)\to\infty]$ for large w^q. If both distribution functions $P(k)$ and $Q_q(w^q)$ are exponential, $R_q[s(q)]$ has also an exponential form. In the case that two distributions obey power laws, the metastrength distribution also obeys a power law with the exponent $\gamma_s(q)$ given by Eqs. (11) and (12), or (13).

It is important to notice that the q dependence of $\gamma_w(q)$ is described by exponents characterizing the original weight distribution $Q_{\rm org}(w)[=Q_1(w)]$. If $Q_{\rm org}(w)$ has the form

$$Q_{\rm org}(w) \propto \begin{cases} w^{-\gamma_w^+} & (w \gg 1), \\ w^{\gamma_w^-} & (w \ll 1), \end{cases} \tag{14}$$

the asymptotic behavior of $Q_q(w^q)$ for $w^q \gg 1$ is proportional to $w^{-\gamma_w^+ +1-q}$ for $q>0$ and $w^{\gamma_w^- +1-q}$ for $q<0$ because of the relation $Q_q(w^q)=Q_{\rm org}(w)|dw/dw^q|$. Therefore, the exponent $\gamma_w(q)$ defined by Eq. (3) is given by $\gamma_w(q)=1+(\gamma_w^+-1)/q$ for $q>0$ and $\gamma_w(q)=1-(\gamma_w^-+1)/q$ for $q<0$. Combining these relations with Eqs. (11)–(13), we have

$$\gamma_s(q) = \begin{cases} \min\left[1-\frac{1}{q}(\gamma_w^-+1), 1-\frac{1}{q}(\gamma_k-1)(\gamma_w^-+1)\right] & (q<-\gamma_w^- -1), \\ \min[\gamma_k, 2] & (q=-\gamma_w^- -1), \\ \min\left[\gamma_k, 1-\frac{1}{q}(\gamma_w^-+1)\right] & (-\gamma_w^- -1<q<0), \\ \gamma_k & (q=0), \\ \min\left[\gamma_k, 1+\frac{1}{q}(\gamma_w^+-1)\right] & (0<q<\gamma_w^+-1), \\ \min[\gamma_k, 2] & (q=\gamma_w^+-1), \\ \min\left[1+\frac{1}{q}(\gamma_w^+-1), 1+\frac{1}{q}(\gamma_k-1)(\gamma_w^+-1)\right] & (q>\gamma_w^+-1). \end{cases} \tag{15}$$

Therefore, we can determine the exponent $\gamma_s(q)$ by three exponents γ_k, γ_w^+, and γ_w^- characterizing the weighted network if weights are distributed independently of network topology.

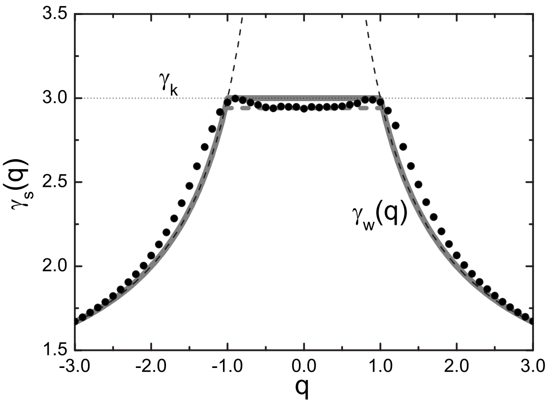

FIG. 1. Metastrength exponent $\gamma_s(q)$ for the BA model with randomly distributed weights as a function of the parameter q. The numerical result (dots) is obtained for 10^2 BA networks with 10^5 nodes. Dotted and dashed lines represent γ_k and $\gamma_w(q)$, respectively. The metastrength exponent $\gamma_s(q)$ predicted by Eq. (15) is shown by thick gray line. Thick dashed line indicates $\gamma_s(q)=2.94$ which is the usually observed numerical value of the degree exponent γ_k for finite networks of the BA model.

III. EXAMPLES

In order to demonstrate the metaweight analysis and confirm the validity of the argument for topology independent weight distributions, we calculate $\gamma_s(q)$ numerically for four examples. The first example is the Barabási-Albert (BA) model [17] with weighted edges. Weights are randomly distributed on edges with a probability distribution $Q_{\text{org}}(w)$. The BA model is a representative growing scale-free network and has the degree exponent $\gamma_k=3$. In this weighted BA model, the strength distribution is correlated with network topology. For example, hubs have large strengths because all weights of many connected links contribute to their strengths. However, the weight distribution itself in the weighted BA model is obviously independent of network topology. Thus, the metastrength exponent $\gamma_s(q)$ can be predicted by Eq. (15). Here we choose the weight distribution function as

$$Q_{\text{org}}(w) = \frac{Q_0 w}{1+w^4}, \tag{16}$$

where Q_0 is the normalization constant. This function behaves as $Q_{\text{org}}(w) \propto w^{-3}$ for $w \gg 1$ and $Q_{\text{org}}(w) \propto w$ for $w \ll 1$. Therefore, the exponents in Eq. (14) are $\gamma_w^+=3$ and $\gamma_w^-=1$. For these exponents, γ_k, γ_w^+, and γ_w^-, Eq. (15) leads to

$$\gamma_s(q) = \begin{cases} 1-\dfrac{2}{q} & (q<-1), \\ 3 & (1 \geqslant q \geqslant -1), \\ 1+\dfrac{2}{q} & (q>1). \end{cases} \tag{17}$$

In order to confirm the validity of the above prediction, we have prepared 10^2 BA scale-free networks with weights. Each network contains 10^5 nodes. The q dependence of the metastrength exponent measured numerically for these networks is plotted by dots in Fig. 1, as well as theoretically predicted line (thick gray line). We employed the maximum likelihood estimation method (MLEM) [31] to evaluate the metastrength exponent $\gamma_s(q)$ for all the examples in this section (namely, dots in Figs. 1–3 and 5). The MLEM has two different versions depending on whether data [$s_i(q)$ in our case] are continuous or discrete. It should be noted that we must use the discrete version of the MLEM for $q=0$ while the continuous version must be employed for $q\neq 0$. The numerical result agrees with the theoretical line. However, a slight discrepancy between numerical data and the theoretical prediction appears around crossing points between γ_k and $\gamma_w(q)$ (dotted and dashed lines). This is because the second and third terms of Eq. (10) almost equally contribute to $\tilde{R}_q(z)$ when γ_k and $\gamma_w(q)$ become close to each other, and then the distribution $R_q[s(q)]$ takes a definite power-law form only for very large $s(q)$. Therefore, it is difficult to determine $\gamma_s(q)$ by numerical calculations for finite-size networks.

The second example is the weighted scale-free network model proposed by Barrat, Barthélemy, and Vespignani (BBV) [32]. Although we scattered weights independently of network topology in the previous model, weights in realistic networks generally vary as network growths and it is not obvious that weight distributions are independent of topology. The BBV model is a model whose weights are tuned during the network growth. This model is constructed by the following way. We start with a complete graph with a small number (m_0) of nodes and edges of the initial weight w_0. At each time step, a new node with m edges having the weight w_0 is added and connected to m existing nodes with the probability $\Pi_i = s_i/\Sigma_j s_j$, where s_i is the strength of the ith node. Then, the strength of node i is increased by an extra amount δ. The extra strength is distributed over all the edges of the node i in the proportion of w_{ij}/s_i. The resulting network after a long time becomes a weighted scale-free network possessing power-law distribution functions of degree, weight, and strength. The exponents γ_k, γ_w, and γ_s of this network are theoretically predicted as $\gamma_k=\gamma_s=(4\delta+3)/(2\delta+1)$ and $\gamma_w=2+1/\delta$. In our example, we choose $\delta=1/2$. Therefore, $\gamma_k=\gamma_s=5/2$ and $\gamma_w^+=4$. The distribution function of w for small weights does not take a power-law form in this model. Thus, the exponent γ_w^- is regarded as infinity. If the distribution of w is independent of topology of this network, the metastrength exponent $\gamma_s(q)$ is given by

$$\gamma_s(q) = \begin{cases} \dfrac{5}{2} & (q \leqslant 2), \\ 1+\dfrac{3}{q} & (q>2). \end{cases} \tag{18}$$

The theoretical line given by Eq. (18) is indicated by thick gray line in Fig. 2. The metastrength exponent $\gamma_s(q)$ calculated numerically for 10^2 BBV networks with 10^5 nodes is plotted by dots. The agreement of numerical data with the thick gray line suggests that the weight distribution is independent of network topology.

In general, if weights on a network distribute independently of network topology, the average strength $\langle s(k)\rangle$ over

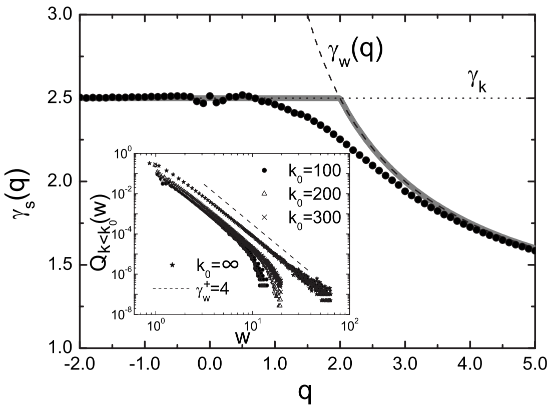

FIG. 2. Metastrength exponent $\gamma_s(q)$ for the BBV model with $\delta=1/2$ and $m=2$ as a function of the parameter q. The numerical result (dots) is obtained for 10^2 BBV networks with 10^5 nodes. Dotted and dashed lines represent γ_k and $\gamma_w(q)$, respectively. The metastrength exponent $\gamma_s(q)$ predicted by Eq. (15) is shown by thick gray line. Slight deviations between numerical results close to $q=0$ and the analytical line are due to our numerical estimation scheme of $\gamma_s(q)$. We employed the continuous version of the MLEM for $q\neq 0$ while data are almost discrete near $q=0$. The inset shows the distribution function $Q_{k<k_0}(w)$ for $k_0=100$, 200, 300, and ∞.

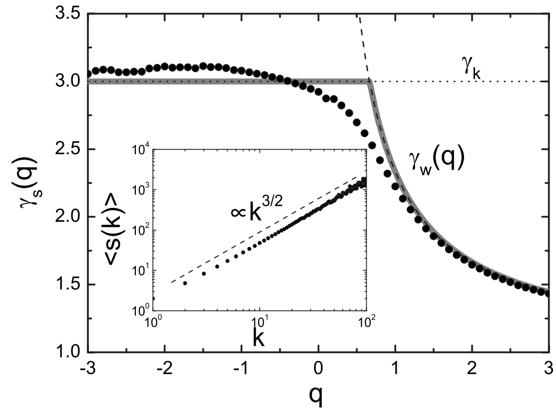

FIG. 3. Metastrength exponent $\gamma_s(q)$ for the PSGN model with $m=1$ and $m'=3$ as a function of the parameter q. The numerical result (dots) is obtained for 200 PSGN's with 10^5 nodes. Dotted and dashed lines represent γ_k and $\gamma_w(q)$, respectively. The metastrength exponent $\gamma_s(q)$ given by Eq. (15) is shown by thick gray line. The inset shows the average strength $\langle s(k)\rangle$ for degree k. The solid line in the inset indicates $\langle s(k)\rangle \propto k^{3/2}$.

nodes of degree k does not change by exchanging weights randomly. This implies that $\langle s(k)\rangle$ keeps its value by replacing w_{ij} by the average weight $\bar{w}$ and then $\langle s(k)\rangle=\bar{w}k$. Since the strength s_i of the ith node is proportional to the degree k_i in the BBV model, the linear relation $\langle s(k)\rangle=\bar{w}k$ holds also for this model. The contrapositive of the above statement deduces that the weight distribution of a weighted network depends on network topology if $\langle s(k)\rangle$ is a nonlinear function of k. However, since the converse proposition is not always true, it is not obvious whether the weight distribution in the BBV model exhibiting the linear relation of $\langle s(k)\rangle$ is independent of network topology. If the distribution function $Q_k(w)$ of weights on edges connected to nodes of degree k depends on k but gives the same average value $\bar{w}$ for different k's, the (total) weight distribution function depends on network topology while $\langle s(k)\rangle=\bar{w}k$. Thus, there is a possibility that the weight distribution depends on topology even if $\langle s(k)\rangle$ is proportional to k. Our results shown in Fig. 2 deny such a possibility for the BBV model. In order to confirm this, we calculated numerically distribution functions $Q_{k<k_0}(w)$ of weights on edges connected to nodes of degrees smaller than k_0. If the distribution function $Q_k(w)$ depends on k, the function $Q_{k<k_0}(w)$ should also depend on k_0. Results of $Q_{k<k_0}(w)$ for the same ensemble of networks with that for Fig. 2 are shown in the inset of Fig. 2. The distribution functions for $k_0=100$, 200, and 300 agree well with the distribution function for the entire set of edges. This clearly shows that weights are scattered in an uncorrelated way to network topology in the BBV model.

Next, we show that the metastrength exponent $\gamma_s(q)$ deviates from Eq. (15) when the weight distribution depends on network topology. Networks with weight distributions correlated to their topology can be obtained by realizing the nonlinear relation $\langle s(k)\rangle \propto k^{\beta}(\beta\neq 1)$. In order to construct such a nonlinear network, we employ the preferential strengthening growing network (PSGN) model proposed by Bianconi [33]. In this model, we start with a set of a few connected nodes with weight w_0 on edges. At each time step, a new node with m edges of weight w_0 attaches to existing m nodes according to the probability $\Pi_i=k_i/\Sigma_j k_j$. At the same time, we choose other m' edges and increase their weights by w_0. The choice of the links is done first by choosing a node with the probability $\Pi_i=s_i/\Sigma_j s_j$ and then by choosing one of the edges of the selected node i with the probability $\Pi_{ij}=w_{ij}/\Sigma_{l\in\nu_i}w_{il}$, where ν_i is the set of nearest neighbor nodes of the ith node. Values of the degree exponent and the weight exponent for the PSGN model are theoretically predicted as $\gamma_k=3$ and $\gamma_w^{+}=(m+2m')/m'$. In this model, the strength of the node i is determined by its degree k_i as

$$s_i \propto \begin{cases} k_i & (m'<m), \\ k_i\ln(k_i) & (m'=m), \\ k_i^{2m'/(m+m')} & (m'>m). \end{cases} \tag{19}$$

Therefore, the exponent $\beta[=\lim_{k\to\infty}\ln\langle s(k)\rangle/\ln k]$ is 1 for $m<m'$ and $2m'/(m+m')$ for $m'>m$. For numerical simulations, we employed $m=1$ and $m'=3$. In this case, $\gamma_w^{+}=7/3$ and $\beta=3/2$. As shown in the inset of Fig. 3, our numerical result indicates the average strength $\langle s(k)\rangle$ proportional to $k^{3/2}$. The metastrength exponent $\gamma_s(q)$ calculated numerically for this model is shown in Fig. 3. In contrast to previous two cases, the profile of $\gamma_s(q)$ for topology-dependent distributions of weights deviates from the theoretical expression given by Eq. (15).

As the final example, we demonstrate the metaweight analysis for a real-world network. The treated system is a correlation network of stock price fluctuations. In this network, nodes represent companies and each company is connected to all other companies, namely, the network forms a complete graph. The weight w_{ij} between the ith and jth com-

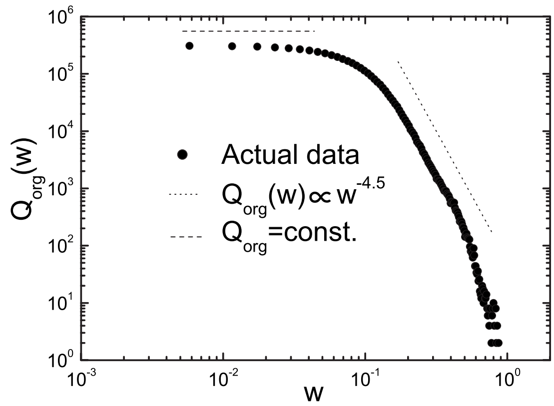

FIG. 4. Weight distribution of the stock price cross-correlation network.

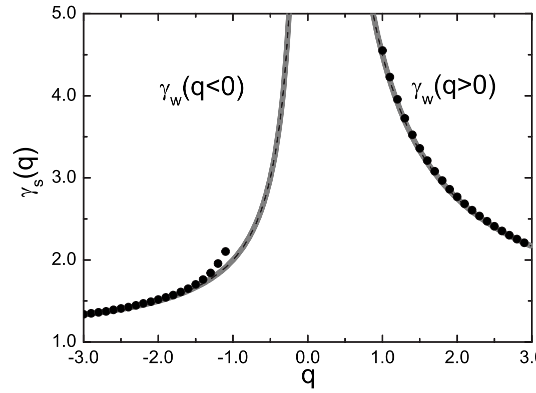

FIG. 5. Metastrength exponent $\gamma_s(q)$ for the stock price cross-correlation network as a function of the parameter q. The numerical result (dots) is obtained for 10^3 companies listed on the First Section of the Tokyo Stock Exchange. Dashed lines represent $\gamma_w(q)$. The metastrength exponent $\gamma_s(q)$ given by Eq. (15) is shown by thick gray lines.

panies is defined as the cross-correlation matrix [34,35], namely,

$$w_{ij} = \frac{\langle G_i G_j\rangle - \langle G_i\rangle\langle G_j\rangle}{\sqrt{(\langle G_i^2\rangle - \langle G_i\rangle^2)(\langle G_j^2\rangle - \langle G_j\rangle^2)}}, \tag{20}$$

where $\langle\cdots\rangle$ means a temporal average over a given time interval T and G_i is the relative return defined by

$$G_i(t) = S_i(t) - \frac{1}{N}\sum_i S_i(t). \tag{21}$$

Here $S_i(t)$ is the stock price change of the ith company after a time δt, that is,

$$S_i(t) = \ln Y_i(t+\delta t) - \ln Y_i(t), \tag{22}$$

and $Y_i(t)$ is the stock price at time t. It has been already reported that the distribution functions of weights and strengths of stock price cross-correlation networks have power-law forms [35].

We examine the cross-correlation network composed of 10^3 companies listed on the First Section of the Tokyo Stock Exchange. The time interval T for the temporal average in Eq. (20) is chosen as T=2 years. Results are averaged over five networks for different time periods from January, 1994 to April, 2005. The resulting weight distribution, Fig. 4, shows a power-law form with $\gamma_w^+ \approx 4.5$ for $w \gg \bar{w}$, while it becomes constant (γ_w^-=0) for $w \ll \bar{w}$. Since the network is a complete graph, the degree exponent γ_k can be treated as infinity and the weight distribution is obviously independent of the network topology. Therefore, Eq. (15) gives

$$\gamma_s(q) = \begin{cases} 1 - \dfrac{1}{q} & (q<0), \\[2ex] 1 + \dfrac{3.5}{q} & (q>0). \end{cases} \tag{23}$$

The numerically calculated metastrength exponent $\gamma_s(q)$ is plotted as a function of q in Fig. 5 with the theoretical line given by Eq. (23). The analytical prediction well describes the numerical result. As shown in this example, a weighted network with complete graph topology should possess the scale-free nature both for positive and negative q's unless γ_w^+ and γ_w^- diverge. These examples in this section clearly demonstrate that the present analysis provides a powerful tool for studying statistical properties of weighted complex networks.

IV. CONCLUSIONS

We have proposed a method to analyze statistical properties of weighted complex networks. Introducing the metaweight defined by w^q, we characterized systematically structural features of networks depending on how strongly weights are emphasized. It has been found that only three exponents are required to characterize the metastrength distribution function $R_q[s(q)]$ for any q if the weight distribution is independent of network topology. In such a case, the exponent characterizing $R_q[s(q)]$ for an arbitrary value of q can be calculated by three exponents γ_k, γ_w^+, and γ_w^-. Furthermore, in order to confirm our analytical arguments, we performed numerical calculations for four examples, namely, the weighted BA model, the BBV model, the PSGN model, and the stock price cross-correlation network as a real-world complex network. These examples show the efficiency of the present method and the validity of our arguments.

In this paper, we concentrated on metastrength distributions of weighted networks. The concept of the metaweight can be applied to many other statistical quantities characterizing weighted networks. Extending the weighted clustering coefficient [6] to the metaweight version, we can define the metaclustering coefficient $C(q)$ by $(1/N)\Sigma_{ijl}a_{ij}a_{il}a_{jl}(w_{ij}^q + w_{il}^q)/[2(k_i-1)s_i(q)]$, where a_{ij} is the (i,j) element of the binary adjacency matrix and $s_i(q)$ is the metastrength defined by Eq. (1). If $C(q<0)$ is larger than $C(q>0)$, the network is densely connected mainly by weak edges and sparsely connected by strong edges, and vice versa. Therefore, the analysis of the metaclustering coefficient reveals detailed relations between network topology and the weight distribution. Another candidate of possible applications of the metaweight analysis is the module decomposition by using metaweights.

Defining the metabetweenness centrality based on shortest distances of metaweighted paths, a weighted network can be divided into communities depending on q. This may make it possible to classify nodes in the network from continuously changing viewpoints by varying the value of q. Recently, it has been reported that the detailed modular structure of a weighted complex network can be elucidated by introducing artificially extended weights [36]. The module decomposition based on metaweights might be an alternative way of such studies. As seen from these possible applications, the concept of the metaweight introduced in this paper has a great potential to analyze weighted complex networks.

ACKNOWLEDGMENTS

We would like to thank N. Masuda and S. N. Dorogovtsev for stimulating discussions. Numerical calculations were performed partially on the HITACHI SR-11000 of Supercomputer Center, Institute for Solid State Physics, University of Tokyo. This work was supported in part by a Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science (Grant No. 19560001) and for the 21st Century COE Program, entitled "Topological Science and Technology," from the Ministry of Education, Culture, Sport, Science and Technology of Japan (MECSST).

[1] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[2] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, and J. Wiener, Comput. Netw. **33**, 309 (2000).
[3] H. Ebel, L.-I. Mielsch, and S. Bornholdt, Phys. Rev. E **66**, 035103(R) (2002).
[4] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, Nature (London) **407**, 651 (2000).
[5] H. Jeong, S. P. Mason, A.-L. Barabási, and Z. N. Oltvai, Nature (London) **411**, 41 (2001).
[6] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101**, 3747 (2004).
[7] M. E. J. Newman, Phys. Rev. E **64**, 016131 (2001).
[8] M. E. J. Newman, Phys. Rev. E **64**, 016132 (2001).
[9] R. Guimerá, S. Mossa, A. Turtschi, and L. A. N. Amaral, Proc. Natl. Acad. Sci. U.S.A. **101**, 7794 (2005).
[10] R. V. Solé and J. M. Montoya, Proc. R. Soc. London, Ser. B **268**, 2039 (2001).
[11] J. Camacho, R. Guimerá, and L. A. Nunes Amaral, Phys. Rev. Lett. **88**, 228102 (2002).
[12] J. A. Dunne, R. J. Williams, and N. D. Martinez, Proc. Natl. Acad. Sci. U.S.A. **99**, 12917 (2002).
[13] J. M. Montoya, S. L. Pimm, and R. V. Sole, Nature (London) **442**, 259 (2006).
[14] G. Bianconi and A.-L. Barabási, Phys. Rev. Lett. **86**, 5632 (2001).
[15] J. P. K. Doye, Phys. Rev. Lett. **88**, 238701 (2002).
[16] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[17] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[18] C. Song, S. Havlin, and H. A. Makse, Nature (London) **433**, 392 (2005).
[19] M. E. J. Newman, Phys. Rev. E **70**, 056131 (2004).
[20] S. H. Yook, H. Jeong, A.-L. Barabási, and Y. Tu, Phys. Rev. Lett. **86**, 5835 (2001).
[21] G. Mukherjee and S. S. Manna, Phys. Rev. E **74**, 036111 (2006).
[22] K. Park, Y.-C. Lai, and N. Ye, Phys. Rev. E **70**, 026109 (2004).
[23] P. J. Macdonald, E. Almass, and A.-L. Barabási, Europhys. Lett. **72**, 308 (2006).
[24] L. A. Braunstein, S. V. Buldyrev, R. Cohen, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **91**, 168701 (2003).
[25] M. Li, D. Wang, Y. Fan, and Z. Di, New J. Phys. **72**, 8 (2006).
[26] T. Nakayama and K. Yakubo, *Fractal Concepts in Condensed Matter Physics* (Springer-Verlag, Berlin, 2003).
[27] S. N. Dorogovtsev and J. F. F. Mendes, e-print arXiv:cond-mat/0408343.
[28] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[29] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks From Biological Nets to the Internet and WWW* (Oxford University, Oxford, 2003).
[30] We call terms depending on γ_k, $\gamma_w(q)$, or $\gamma_s(q)$ nonanalytic terms regardless of whether these exponents are noninteger or not.
[31] A. Clauset, C. R. Shalizi, and M. E. J. Newman, e-print arXiv:0706.1062v1.
[32] A. Barrat, M. Barthélemy, and A. Vespignani, Phys. Rev. Lett. **92**, 228701 (2004).
[33] G. Bianconi, Europhys. Lett. **71**, 1029 (2005).
[34] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, 2007).
[35] H.-J. Kim, Y. Lee, B. Kahng, and I.-M. Kim, J. Phys. Soc. Jpn. **71**, 2133 (2002).
[36] A. Arenas, A. Fernández, and S. Gómez, New J. Phys. **10**, 053039 (2008).